APPLICATION OF THE METHOD OF SPATIAL CHARACTERISTICS TO THE SOLUTION OF AXIALLY SYMMETRIC PROBLEMS RELATING TO THE PROPAGATION OF ELASTIC WAVES

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We present a difference scheme, based on the method of spatial characteristics, for solving axially symmetric dynamical problems of the theory of elasticity. Consideration is given to the possibility of solving a Cauchy problem and a problem for a solid or a hollow cylinder which takes boundary conditions into account. It is suggested that linear problems may be solved by this method. An example is given in which the parameters characterizing the stress-deformation state of a semiinfinite cylinder are calculated, the points of the end of the cylinder being given an initial axial velocity. The calculation of these parameters was carried out on the BÉSM-6 computer.

In [1] a finite-difference method was used to treat the initial stage of the impact involving the axially symmetric elastic collision of two circular plates made of the same material. The set of dynamical equations, written in terms of displacements, was integrated up to the time instant at which the distance travelled by the longitudinal wave is less than a tenth of the plate diameter.

An analytical treatment was given in [2] of the problem involving the collision of two elastic cylinders in which Laplace integral transforms in the time and Fourier transforms in the space coordinate were used. Inversion of the resulting transforms was accomplished for large values of the transform parameters, which corresponds to the construction of an asymptotic solution valid for small time intervals following the instant of impact. No numerical analysis of the results obtained was given.

D. S. Butler presented in [3] a numerical scheme for solving a hyperbolic system of equations depending on three variables.

A difference scheme applicable to the planar two-dimensional problem of the dynamic theory of elasticity, based on the method of spatial characteristics, was presented in [4] to solve a Cauchy problem.

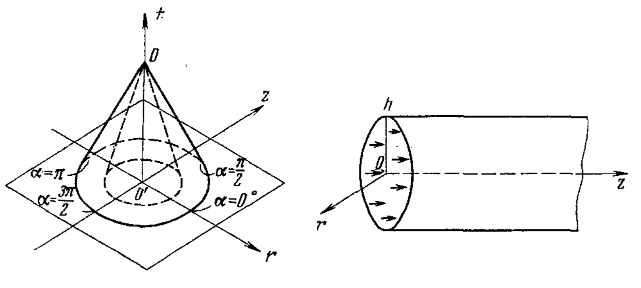

Fig. 1

The dynamics of elastoplastic waves for a sectorial ring was presented by the author in [5] using a finite difference method. Distributions were given for the radial and circumferential components of the stress.

In [6] the method of spatial characteristics was used to study the planar two-dimensional problem concerning propagation and diffraction of elastic waves in a halfstrip of finite width. Calculations of the kinematic and dynamic characteristics of the elastic field were presented in graphical form.

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We present below a general scheme for the spatial characteristics method applicable to axially symmetric problems of the dynamic linear theory of elasticity.

1. Method of Solution. In an r, z, θ system of cylindrical coordinates the equations of motion for the spatial axially symmetric case may be written, in the usual notation, as follows:

$$\frac{\partial\sigma_{rr}}{\partial r}+\frac{\partial\tau_{rz}}{\partial z}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=\rho\frac{\partial^2 u_r}{\partial t^2}$$
$$\frac{\partial\tau_{rz}}{\partial r}+\frac{\partial\sigma_{zz}}{\partial z}+\frac{\tau_{rz}}{r}=\rho\frac{\partial^2 u_z}{\partial t^2} \tag{1.1}$$

To the Eqs. (1.1) there must be adjoined the Hooke's Law relations

$$\sigma_{rr}=\lambda\theta+2\mu\frac{\partial u_r}{\partial r},\quad \sigma_{zz}=\lambda\theta+2\mu\frac{\partial u_z}{\partial z}$$
$$\theta=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z},\quad \sigma_{\theta\theta}=\lambda\theta+2\mu\frac{u_r}{r},\quad \tau_{rz}=\mu\left(\frac{\partial u_r}{\partial z}+\frac{\partial u_z}{\partial z}\right) \tag{1.2}$$

in which $u_r(r, z, t)$ and $u_z(r, z, t)$ denote, respectively, the r and z components of the elastic displacement vector and λ and μ are the Lamé parameters.

We now transform the set of Eqs. (1.1) and (1.2) so that the functions appearing in the transformed system will have a direct physical meaning. We accomplish this by differentiating Hooke's Law with respect to the time after adding and subtracting expressions for the stresses σ_{rr} and σ_{zz}.

Proceeding in this way and then introducing independent variables and functions through the relations

$$r^\circ=\frac{r}{h},\quad z^\circ=\frac{z}{h},\quad t^\circ=\frac{at}{h},\quad p=\frac{\sigma_{rr}+\sigma_{zz}}{2\rho a^2}$$
$$q=\frac{\sigma_{rr}-\sigma_{zz}}{2\rho a^2},\quad \tau=\frac{\tau_{rz}}{\rho a^2},\quad \sigma=\frac{\sigma_{\theta\theta}}{\rho a^2},\quad u^\circ(r^\circ, z^\circ, t^\circ)=a^{-1}\frac{\partial u_r}{\partial t}$$
$$v^\circ(r^\circ, z^\circ, t^\circ)=a^{-1}\frac{\partial u_z}{\partial t},\quad a=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad b=\sqrt{\frac{\mu}{\rho}}$$
$$\gamma=\frac{a}{b}>1 \tag{1.3}$$

we rewrite Eqs. (1.1) and (1.2) in the following equivalent form (the lower subscript denotes partial differentiation):

$$u_t-p_r-q_r-\tau_z=(p+q-\sigma)/r$$
$$v_t-p_z+q_z-\tau_r=\frac{\tau}{r}$$
$$\gamma^2(\gamma^2-1)^{-1}p_t-u_r-v_z=(\gamma^2-2)(\gamma^2-1)^{-1}u/r$$
$$\gamma^2 q_t-u_r+v_z=0$$
$$\gamma^2(\gamma^2-1)^{-1}\sigma_t-u_r-v_z=\gamma^2(\gamma^2-2)^{-1}u/r$$
$$\gamma^2\tau_t-u_z-v_r=0 \tag{1.4}$$

We thus obtain a system of six first order equations, with variable coefficients, in the six unknown functions u, v, p, q, σ, and τ; here and in what follows the degree sign superscript on the independent and dependent variables will be omitted for simplicity. The system (1.4) depends on one material parameter γ.

We make the following transformation in order that the system (1.4) takes on a symmetric form. We replace the fifth equation by the difference between the fifth and third equations. From the latter we determine u/r and substitute it into the third equation. The matrix of the resulting system is symmetric with respect to t, r, z:

$$u_t-p_r-q_r-\tau_z=r^{-1}(p+q-\sigma)$$
$$v_t-p_z+q_z-\tau_r=r^{-1}\tau$$
$$\gamma^4(3\gamma^2-4)^{-1}p_t+\gamma^2(2-\gamma^2)(3\gamma^2-4)^{-1}\sigma_t-u_r-v_z=0$$
$$\gamma^2 q_t-u_r+v_z=0$$
$$\gamma^2(2-\gamma^2)(3\gamma^2-4)^{-1}p_t+\gamma^2(\gamma^2-1)(3\gamma^2-4)^{-1}\sigma_t=r^{-1}u$$
$$\gamma^2\tau_t-u_z-v_r=0 \tag{1.5}$$

The surface $\Phi(r, z, t)$ = const will be a characteristic surface of the system (1.5) if it satisfies the following differential equation:

$$[\Phi_t^2-\Phi_r^2-\Phi_z^2][\Phi_t^2-(\Phi_r^2+\Phi_z^2)\gamma^{-2}]\Phi_t^2\gamma^2(\gamma^2-2)^{-1}=0 \tag{1.6}$$

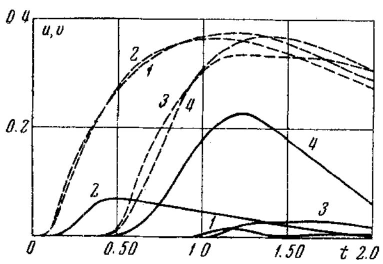

Fig. 2

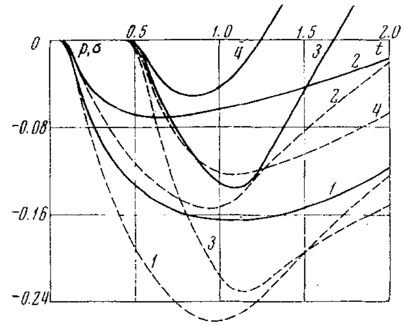

Fig. 3

Equation (1.6) yields two families of circular cones, the tangents of whose half angles, formed with the t axis, are, respectively, 1 and $1/\gamma$. The equation $\Phi_t = 0$ yields the cone axes [4]. The geometry of the cones is shown in Fig. 1a.

Following standard procedure we obtain two relations for the external and internal cones, respectively:

$$\cos\alpha\, du + \sin\alpha\, dv + dp + \cos 2\alpha\, dq + \sin 2\alpha\, d\tau = -S_1(\alpha)\, dt$$
$$-\sin\alpha\, du + \cos\alpha\, dv - \gamma \sin 2\alpha\, dq + \gamma\cos 2\alpha\, d\tau = -S_2(\alpha)\, dt \tag{1.7}$$

$$S_1(\alpha) = (2\gamma^{-2} - 1)\sin^2\alpha\, u_r + (2^{-1} - \gamma^{-2})\sin 2\alpha\, u_z + (2^{-1} - \gamma^{-2})$$
$$\times \sin 2\alpha\, v_r + (2\gamma^{-2} - 1)\cos^2\alpha\, v_z - 2\sin^2\alpha\cos\alpha\, q_r + 2\cos^2\alpha\sin\alpha\, q_z$$
$$+ \sin\alpha\cos 2\alpha\, \tau_r - \cos\alpha\cos 2\alpha\, \tau_z - \cos\alpha\,(p + q - \sigma)\, r^{-1} - \sin\alpha\, \tau\, r^{-1} - (1 - 2\gamma^{-2})\, u r^{-1} \tag{1.8}$$

$$S_2(\alpha) = 2^{-1}\gamma^{-1}\sin 2\alpha\, u_r - \gamma^{-1}\cos^2\alpha\, u_z + \gamma^{-1}\sin^2\alpha\, v_r - 2^{-1}\gamma^{-1}$$
$$\times \sin 2\alpha\, v_z + \sin\alpha\, p_r - \cos\alpha\, p_z - \cos 2\alpha\sin\alpha\, q_r + \cos 2\alpha\cos\alpha\, q_z$$
$$-2\sin^2\alpha\cos\alpha\, \tau_r + 2\cos^2\alpha\sin\alpha\, \tau_z + \sin\alpha\,(p + q - \sigma)\, r^{-1} - \cos\alpha\, \tau r^{-}$$

In the Eqs. (1.6) and (1.8) α $(0 \le \alpha \le 2\pi)$ denotes an arbitrary parameter.

We show now how it is possible to solve a Cauchy problem for the system (1.5), i.e., how it is possible to determine all six functions at the point O, knowing them at an arbitrary point of the plane t=0 (the plane of values of the initial data). We integrate Eqs. (1.7) and (1.8) along an arbitrary generator (a bicharacteristic) of each cone from the point O to its intersection with the rz-plane (the position of the generator on the circular cones is fixed by assigning the parameter α, for example, $\alpha = \alpha_i$)

$$\delta u \cos\alpha_i + \delta v \sin\alpha_i + \delta p + \delta q \cos 2\alpha_i + \delta\tau \sin 2\alpha_i = -{}^1/_2\, k\, [S_1(\alpha_i)_i + S_1(\alpha_i)_O] - W_1(\alpha_i) + O(k^3) \tag{1.9}$$

$$-\delta u \sin\alpha_i + \delta v \cos\alpha_i - \delta q\, \gamma \sin 2\alpha_i + \delta\tau\gamma\cos 2\alpha_i = -{}^1/_2 k\, [S_2(\alpha_i)_i + S_2(\alpha_i)_O] - W_2(\alpha_i) + O(k^3) \tag{1.10}$$

where for brevity we have introduced the notation

$$W_1(\alpha_i) = (u' - u_i)\cos\alpha_i + (v' - v_i)\sin\alpha_i + (p' - p_i) + (q' - q_i)\cos 2\alpha_i + (\tau' - \tau_i)\sin 2\alpha_i$$
$$W_2(\alpha_i) = -(u' - u_i)\sin\alpha_i + (v' - v_i)\cos\alpha_i - \gamma(q' - q_i)\sin 2\alpha_i + (\tau' - \tau_i)\gamma\cos 2\alpha_i.$$

In Eqs. (1.9) and (1.10) k denotes the time step $k = \Delta t = OO'$; the subscript i means that the corresponding function is calculated at the point where the bicharacteristic $\alpha = \alpha_i$ of the internal and external cone intersects the plane of the initial data, t=0; the subscript 0 identifies the value of the function at the point O, the prime means that the function is calculated at the point O', δu is the increment $u_0 - u'$, etc.

We integrate the initial set of Eqs. (1.5) along a cone axis. We then obtain, using, for example, the first equation,

$$\delta u = {}^1/_2 k\, [(p_r + q_r + \tau_z + r^{-1}(p + q - \sigma))_O + (p_r + q_r + \tau_z + r^{-1}(p + q - \sigma))'] + O(k^3) \tag{1.11}$$

and five similar equations.

In Eqs. (1.9) and (1.10), for the sake of convenience, we take the values of α_i equal to 0, $\pi/2$, π, and $3\pi/2$, thereby fixing four bicharacteristics on the internal and external cones. As a result, we obtain a set of eight linear algebraic equations, which when supplemented by six relations of the type (1.11) yields a set of fourteen equations. We must eliminate from this system the derivatives of the six functions at the point O at which they are unknown.

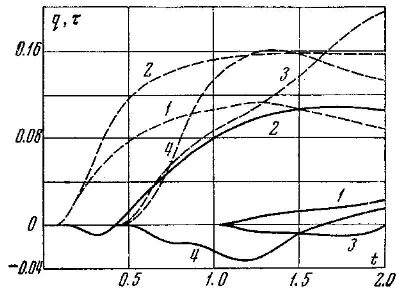

Fig. 4

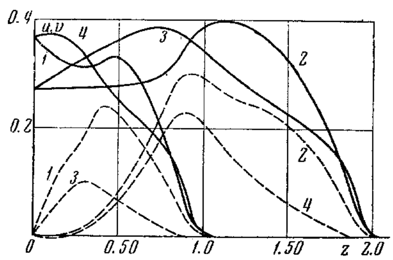

Fig. 5

A difference scheme, suitable for use on an electronic digital computer, can be obtained in the following way. We denote the relations obtained for the external cone by using Eq. (1.9) on the four bicharacteristics $\alpha_i = 0,\ \pi/2,\ \pi,\ 3\pi/2$ by the digits 1, 2, 3, and 4, respectively. The relations obtained for the internal cone using Eq. (1.10) with these same α_i values will be denoted by the digits 5, 6, 7, and 8, respectively. Subtracting relation 6 from relation 8, we obtain

$$2\delta u = -{}^1/_2 kS_2\left(\frac{3\pi}{2}\right)_i - W_2\left(\frac{3\pi}{2}\right) + {}^1/_2 kS_2\left(\frac{\pi}{2}\right)_i + W_2\left(\frac{\pi}{2}\right) + {}^1/_2 k\,[2\,(p_r + q_r + r^{-1}(p+q-\sigma))]_O \qquad (1.12)$$

By subtracting relation (3) from relation (1) we obtain

$$2\delta u = -\frac{k}{2}S_1(0)_i - W_1(0) + \frac{k}{2}S_1(\pi) + W_1(\pi) \ + \frac{k}{2}\{2\,[\tau_z + r^{-1}(p+q-\sigma)]\}_O \qquad (1.13)$$

From the relations (1.12) and (1.13) we obtain the derivatives $^1/_2 k(p_r+q_r)_O$ and $^1/_2 k(\tau_z)_O$ at the point O, which makes it possible to eliminate them from the right side of Eq. (1.11). Similarly we can eliminate the derivatives in the other five equations of the type (1.11), obtaining thereby a set of difference equations for determining the six unknown functions u, v, p, q, σ, and τ at the point O

$$\begin{gathered}
\delta u - \frac{k}{2r}\delta p + \frac{k}{2r}\delta\sigma - \frac{k}{2r}\delta q = \frac{a_1 - a_3}{2} - \frac{b_2 - b_4}{2} - {}^1/_2 k\,(p_r + q_r + \tau_z)' \\
\delta v - \frac{k}{2r}\delta\tau = \frac{a_2 - a_4}{2} + \frac{b_1 - b_3}{2} - {}^1/_2 k\,(p_z - q_z + \tau_r)' \\
\gamma^2(\gamma^2-1)^{-1}\delta p - (\gamma^2-2)(\gamma^2-1)^{-1}\frac{k}{2r}\delta u = \frac{a_1 + a_2 + a_3 + a_4}{2} - (\gamma^2-2)\gamma^{-2}\frac{k}{2}(u_r + v_z)' + (\gamma^2-2)(\gamma^2-1)^{-1}\gamma^{-2}\frac{k}{r}u' \qquad (1.14) \\
\gamma^2\delta q = \frac{a_1 - a_2 + a_3 - a_4}{2} + (\gamma^2-2)\gamma^{-2}\frac{k}{2}(u_r - v_z)' \\
\gamma^2(\gamma^2-1)(3\gamma^2-4)^{-1}\delta\sigma + \gamma^2(2-\gamma^2)(3\gamma^2-4)^{-1}\delta p - \frac{k}{2r}\delta u = kr^{-1}u' \\
\gamma\delta\tau = \frac{b_1 - b_2 + b_3 - b_4}{2} - {}^1/_2 k\gamma^{-1}(u_z + v_r)'
\end{gathered}$$

where

$$a_i = -{}^1/_2 kS_1(\alpha_i)_i - W_1(\alpha_i), \qquad b_i = -{}^1/_2 kS_2(\alpha_i)_i - W_2(\alpha_i)$$
$$\alpha_i = (i-1)\,\pi/2, \qquad i = 1, 2, 3, 4$$

The set of six equations (1.14) enables us to express the six functions at the point O in terms of the initial values of these functions and their derivatives, given at the point O' of the initial data plane and at eight neighboring points of this plane. By displacing the point O' in the plane t = 0 it is possible to obtain the values of the functions in a plane parallel to the plane t = 0 and formed by the vertices of the circular characteristics of the cones. This plane is at a distance of $k = \Delta t$ from the initial data plane. Proceeding further in a similar way and choosing Δt, Δr, and Δz sufficiently small, we can obtain all six functions involved in an axially symmetric problem of the linear dynamic theory of elasticity for all r, z, and t.

In concluding this section we remark that the system (1.14) reduces to a coupled system in five increments for the functions u, p, v, q, and τ, which differs from the case of a two-dimensional problem by terms of order k/r. The increment for σ is a linear combination of the increments for p and u.

2. Impact on the Endface of a Semiinfinite Cylinder. We assume that a semiinfinite circular elastic cylinder of radius h (composed of a homogeneous and isotropic material) is subjected at time t = 0 at points of its endface to a known action (Fig. 1b). The problem to be investigated is that of determining the parameters of the stress-deformation state in the domain $0 \le r \le h$, $z \ge 0$ for $t > 0$, assuming that the speed of the longitudinal wave is equal to a, the speed of the transverse wave is equal to b, and the material density is ρ.

In dimensionless form the boundary-value problem reduces to integrating over the domain $0 \le r \le 1$, $0 \le z < \infty$, $t > 0$ the system of Eqs. (1.5) for the zero initial data

$$u = v = p = q = \sigma = \tau = u_t = v_t = p_t = q_t = \sigma_t = \tau_t = 0 \quad \text{for} \quad t \leqslant 0 \tag{2.1}$$

and the following boundary conditions:

$$p + q = 0, \quad \tau = 0 \quad \text{for} \quad r = 1, \quad 0 \leqslant z < \infty, \quad t > 0 \tag{2.2}$$

$$v = v_0(t), \quad u = 0 \quad \text{for} \quad z = 0, \quad 0 \leqslant r \leqslant 1, \quad t > 0 \tag{2.3}$$

where $v_0(t)$ is an arbitrarily prescribed function. Initially the cylinder is in an undistrubed state, its lateral surface is stress-free, and the particle velocity vector is prescribed at its end-face (other types of conditions can also be prescribed). We solve this problem numerically, the method used being essentially that of Section 1.

The system (1.14) allows us to make calculations at an arbitrary interior point of the cylinder. At boundary points it cannot be used directly since some of the bicharacteristics pass outside the domain in question and intersect the initial data plane $t = 0$ at points where the solution is not defined. At the endface $z = 0$, $0 \le r \le 1$, the terms a_4, b_4, appearing in Eqs. (1.14), may be determined in terms of points located to the left of the plane $z = 0$. Eliminating $a_4 b_4$ from Eqs. (1.14), we obtain a set of four equations in the six increments; however, when these equations are supplemented by the boundary conditions (2.3), the resulting system becomes a closed system, solvable at an arbitrary point of the endface, not an angular point.

We integrated the initial set of equations with an accuracy to $O(k^3)$. If the set of Eqs. (1.14) is written out in detail, it is possible to obtain expressions of the following types:

$$\begin{gathered} f(r + ck, z) - f(r - ck, z) \\ f(r + ck, z) + f(r - ck, z) - 2f(r, z) \\ k\,[f_r(r, z + ck) - f_r(r, z - ck)] \\ k\,[f_r(r, z + ck) + f_r(r, z - ck) - 2f_r(r, z)] \end{gathered} \tag{2.4}$$

where $c = 1$ and $1/\gamma$ for the internal and external cones, respectively. Expansion of the function f in a Taylor's series in a neighborhood of the point (r, z) shows that the expressions (2.4) differ, respectively, from the following quantities by terms of order $O(k^3)$:

$$2ckf_r(r, z), \quad (ck)^2 f_{rr}(r, z), \quad 2ck^2 f_{rz}(r, z), \quad 0 \tag{2.5}$$

The necessary accuracy will have been attained when the difference approximation for the first and second partial derivatives at the point (r, z) is of order $O(k^2)$ and $O(k)$, respectively. For interior points of the domain it is convenient to use central differences; for boundary points, – forward and backward difference approximations. After the transformations guaranteeing an accuracy of order $O(k^3)$ have been made, it is convenient to rewrite the system (1.14) in the following form:

$$\delta u - \frac{k}{2r}\delta p + \frac{k}{2r}\delta\sigma - \frac{k}{2r}\delta q = \frac{k^2}{2}(1 - \gamma^{-2})\, v_{rz} + \frac{k^2}{2}(u_{rr} + \gamma^{-2}u_{zz})$$
$$+ \frac{k^2}{2r}(1 - 2\gamma^{-2})\, u_r + k\left(\frac{p + q - \sigma}{r}\right)' + k\,(p_r + q_r + \tau_z)$$

$$\delta v - \frac{k}{2r}\delta\tau = \frac{k^2}{2}(1 - \gamma^{-2})\, u_{rz} + \frac{k^2}{2}(v_{zz} + \gamma^{-2}v_{rr}) + \frac{k^2}{2r}(1 - 2\gamma^{-2})\, u_z + \frac{k}{r}\tau' + k\,(p_z - q_z + \tau_r)$$

$$\gamma^2(\gamma^2 - 1)^{-1}\delta p - (\gamma^2 - 2)(\gamma^2 - 1)^{-1}\frac{k}{2r}\delta u = \frac{k^2}{2}(2\tau_{rz} + p_{zz} + p_{rr} + q_{rr} - q_{zz})$$
$$+ k\,(u_r + v_z) + k\,(\gamma^2 - 2)(\gamma^2 - 1)^{-1}\frac{u'}{r} + \frac{k^2}{2r}(p_r + q_r - \sigma_r) + \frac{k^2}{2r}\tau_z$$

$$\gamma^2\delta q = \frac{k^2}{2}(p_{rr} + q_{rr} - p_{zz} + q_{zz}) + k\,(u_r - v_z) + \frac{k^2}{2r}(p_r + q_r - \sigma_r) - \frac{k^2}{2r}\tau_z$$

$$\gamma^2(\gamma^2 - 1)(3\gamma^2 - 4)^{-1}\delta\sigma + \gamma^2(2 - \gamma^2)(3\gamma^2 - 4)^{-1}\delta p - \frac{k}{2r}\delta u = \frac{k}{r}u'$$

$$\gamma^2\delta\tau = \frac{k^2}{2}(2p_{rz} + \tau_{rr} + \tau_{zz}) + k\,(u_z + v_r) + \frac{k^2}{2r}(\tau_r + p_z + q_z - \sigma_z)$$

At an arbitrary point of the endface $z = 0$, $0 \le r < 1$, not an angular point, calculations are made using the system of equations

$$\left[1 + \left(\frac{k}{2r}\right)^2 (3\gamma^2 - 4)\,\gamma^{-2}(\gamma^2 - 1)^{-1}\right]\delta u - \frac{k}{2r}(\gamma^2 - 1)^{-1}\delta p - \frac{k}{2r}\delta q$$
$$+ \gamma\delta\tau + \left(\frac{k}{2r}\right)^2 2\,(3\gamma^2 - 4)(\gamma^2 - 1)^{-1}\gamma^{-2}u' = \frac{k^2}{2}(1 - \gamma^{-2})\, v_{rz}$$

$$+\frac{k^2}{2}(u_{rr}+2\gamma^{-1}p_{rz}+\gamma^{-1}\tau_{rr})+k(p_r+q_r+\tau_z)+k\gamma^{-1}(v_r+u_z)$$
$$+\frac{k^2}{2r}(1-2\gamma^{-2})u_r+\gamma^{-1}(\tau_r+p_z+q_z-\sigma_z)+\frac{k}{r}(p+q-\sigma)'$$
$$\delta v-(\gamma^2-2)(\gamma^2-1)^{-1}\frac{k}{2r}\delta u+\gamma^2(\gamma^2-1)^{-1}\delta p-\frac{k}{2r}\delta\tau=\frac{k^2}{2}(1-\gamma^{-2})u_{rz}$$
$$+\frac{k^2}{2}(2\tau_{rz}+p_{rr}+q_{rr}+\gamma^{-2}v_{rr})+k(p_z-q_z+\tau_r)+k(u_r+v_z)$$
$$+\frac{k^2}{2r}(p_r+q_r-\sigma_r+\tau_z)+\frac{k^2}{2r}(1-2\gamma^{-2})u_z+\frac{k}{r}\tau'+\frac{k}{r}(\gamma^2-2)(\gamma^2-1)^{-1}u'$$
$$\gamma^2(\gamma^2-1)^{-1}\delta p-(\gamma^2-2)(\gamma^2-1)^{-1}\frac{k}{2r}\delta u+\gamma^2\delta q=k^2(\tau_{rz}+p_{rr}+q_{rr})$$
$$+\frac{k}{r}(p_r+q_r+\sigma_r)+\frac{k}{r}(\gamma^2-2)(\gamma^2-1)^{-1}u'+2ku_r$$
$$-\frac{k}{2r}\delta u+\gamma^2(2-\gamma^2)(3\gamma^2-4)^{-1}\delta p+\gamma^2(\gamma^2-1)(3\gamma^2-4)^{-1}\delta\sigma=\frac{k}{r}u'$$

In an analogous way we can obtain a solvable system of equations for calculating conditions at points of the free surface $r=1$, $0\leq z<\infty$. This system can be obtained by eliminating the values a_1 and b_1 from the system (1.14) and appending the boundary conditions for $r=1$, $z>0$.

For the angular point $r=1$, $z=0$ the system (1.14) may be solved by eliminating a_1, b_1, a_4, b_4 (since the corresponding cones pass outside the domain in question) and adjoining the boundary conditions at the lateral surface $r=1$ and on the endface $z=0$.

We obtain, as a result, the set of two equations

$$-\frac{k}{2r}\delta u+\gamma^2(2-\gamma^2)(3\gamma^2-4)^{-1}\delta p+\gamma^2(\gamma^2-1)(3\gamma^2-4)^{-1}\delta\sigma=\frac{k}{r}u'$$
$$\left[1+(\gamma^2-2)(\gamma^2-1)^{-1}\frac{k}{2r}\right]\delta u-\delta v-\left[\frac{k}{2r}+\gamma^2(\gamma^2-1)^{-1}\right]\delta p-\frac{k}{2r}\delta q$$
$$+\frac{k}{2r}\delta\sigma+\left(\gamma+\frac{k}{2r}\right)\delta\tau=-\frac{k^2}{2}(1-\gamma^{-2})(u_{rz}-v_{rz})-\frac{k^2}{2}(2\tau_{rz}-2\gamma^{-1}p_{rz})$$
$$+\frac{k^2}{2r}(1-2\gamma^{-2})(u_r-u_z)+k(p_r+q_r+\tau_z)-k(p_z-q_z+\tau_r)$$
$$+k\gamma^{-1}(u_z+v_r)-k(u_r+v_z)+\frac{k}{r}(p+q-\sigma)'-\frac{k}{r}(\gamma^2-2)(\gamma^2-1)^{-1}u'$$
$$-\frac{k}{r}\tau'-\frac{k^2}{2r}(p_r+q_r-\sigma_r-\gamma^{-1}p_z-\gamma^{-1}q_z+\gamma^{-1}\sigma_z)-\frac{k^2}{2r}(\tau_z-\gamma^{-1}\tau_r)$$

to which must be adjoined the boundary conditions for $r=1$ and $z=0$. As a result we obtain a system of six equations for determining the six increments at the angular point.

The calculational procedure for a hollow cylinder also requires setting up the equations for an angular point at a free internal surface; this is done in a manner analogous to that described above. The system of equations in this case does not contain singularities for $r\to 0$.

In the case of a solid cylinder, the Eqs. (1.5) contain terms with the factor $1/r$; however, when $r=0$, the numerator in these terms is also equal to zero since u and τ are odd functions of r and since $\sigma_{rr}=\sigma_{\theta\theta}$ for $p+q-\sigma$; i.e., these terms have an indeterminacy of the form 0/0. Resolution of this indeterminacy by l'Hopital's rule yields a limit of zero, since the derivative with respect to r of the functions p, q, and σ vanishes as $r\to 0$; this latter follows from the fact that each of these functions is an even function of r and has a continuous derivative with respect to r in a neighborhood of the point $r=0$.

From physical considerations it follows that on the axis $r=0$ the system of equations for the planar problem and the system of equations for the axially symmetric problem must give the same solution. One can show that the limit of the functions u_r and τ_r as $r\to 0$ is also equal to zero; by the same token, it may be assumed that the indeterminacy in Eqs. (1.5) is of a purely mathematical nature.

We present an example of the calculations. Using the method described above, we integrated the set of Eqs. (1.5), subject to the conditions (2.1)-(2.3), for the following values of the initial data:

$$\gamma=1.87,\ v_0(t)=te^{-t}\ \text{for}\ t\geqslant 0,$$
$$\Delta r=\Delta z=0.025,\ \Delta t=0.0125,\ \Delta t/\Delta r=0.5$$

Some of the calculated results are displayed graphically in Figs. 2-5. Variation of the particle velocities u (solid curves) and v (dashed curves) with the time at the four fixed points with the coordinates ($r=0.1$, $z=0.1$), ($r=0.9$, $z=0.1$), ($r=0.1$, $z=0.5$), ($r=0.9$, $z=0.5$) is shown in Fig. 2 (curves 1, 2, 3, 4, respectively). The lower index denotes the coordinate r, the upper the coordinate z. The values of the transverse velocity u at the points 2 and 4 are of an order larger than the analogous values at the points 1 and 3; this may be explained by the proximity of the latter points to the axis where the motion is quasi-one-dimensional. The transverse and longitudinal velocities at the point 4 are of the same order.

The stresses p (dashed curves) and σ (solid curves) at these same points are shown in Fig. 3, where it is evident that p and σ at the points 1 and 3 are approximately twice as large as the stresses at the points 2 and 4. Variation of the stresses q (dashed curves) and τ (solid curves) with time at the fixed points is shown in Fig. 4. The influence of the diffracting waves is evident graphically on the curve for τ at the point 4, the influence of diffraction at the point 3 is insignificant owing to wave interference.

In Fig. 5 velocity profiles are given for u (dashed curves) and v (solid curves) for (r, t) values given, respectively, by (1.0, 1.0), (1.0, 2.0), (0.8, 1.0) and (0.8, 2.0) (curves 1, 2, 3, 4 on the figure).

The values of the velocities at the cross-sections indicated are comparable among themselves and the nature of their variation can be determined to a significant degree by the boundary conditions at the endface with the phenomenon of diffraction taken into account.

When the results of these calculations are compared with analogous results for the planar problem (a half strip with assigned velocities at its end), a qualitative agreement is found to exist between the basic parameters of the stress-deformation states for the semiinfinite circular cylinder and the half strip.

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